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Abstract

We construct an index theorem for smooth infinite economies that shows that generically the number of equilibria is odd. As a corollary, this gives a new proof of existence and gives conditions that guarantee global uniqueness of equilibria.
Keywords:Uniqueness; determinacy; equilibria; infinite economy; Fredholm map; equilibrium manifold; index theorem; Z-Rothe vector field.
JEL Classification: D50, D51, D80, D90.

Resumen

Construimos un teorema de índice para economías infinitas suaves que demuestra que genéricamente el número de equilibrios es impar. Como corolario, este resultado nos da una nueva demostración de existencia y proporciona condiciones que garantizan unicidad global de equilibrios.
Palabras Clave: Unicidad; determinación; equilibrios; economía infinita; mapeos de Fredholm; variedad de equilibrio; teorema de índice; campo vectorial Z-Rothe.

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1 Introduction

Models of competitive markets may have a consumption space of infinite dimensions, usually arising from models parameterized by time and uncertainty. Within this setting, several authors have addressed the problem of studying if equilibrium prices are locally unique, including Kehoe et al. (1989a), Kehoe et al. (1989b, 1990), Balasko (1997a,c), Chichilnisky and Zhou (1998), Shannon (1999), and Shannon and Zame (2002). Through these results, it has become clear that in order to study determinacy there is always a trade-off between the generality of the consumption space, the generality of utility functions, and the existence and differentiability of individual demand functions. Nevertheless, Shannon and Zame (2002) have studied determinacy in enough generality as to consider this questions almost closed, at least for models of markets where the Negishi approach is permitted, i.e., where the first welfare theorem holds.

In spite of the general determinacy results, an area that still remains largely unexplored is that of counting the number of equilibria. When the consumption space is finite dimensional, Dierker (1972) gave the first solution to this problem by constructing an index theorem that showed, among other things, that the number of equilibria is generically odd.[1] He does this by interpreting the excess demand function as a vector field on the normalized space of prices, and noticing that equilibria are the zeros of this vector field. He defines the notion of index of an equilibrium price system and, using the Poincaré-Hopf Theorem, he shows that the sum of these indices is constant and equal to 1. Since the number of equilibria is odd, in particular it cannot be zero and hence Dierker's index theorem gives a new proof of existence of equilibria. Additionally, if the index at each equilibrium price is greater than zero, then the index theorem also gives conditions for *global*, not just local, uniqueness of equilibria.

Using index theorems to study questions of existence, stability, the number of equilibria, and global uniqueness, has a long tradition throughout the economic literature relying frequently on the arguments provided by Dierker (1972) and Mas-Colell (1985). For example, index theorems have been constructed by Kehoe (1980, 1983) for production economies, Jouini (1992) for

[1]But also see Balasko (1975b, 1988), Dierker (1982) and Varian (1975).

nonconvex production economies, Giraud (2001) for production economies with increasing returns and Mandel (2008) for production economies with externalities. Another area where this approach has been fruitful is in economies with incomplete financial markets, for example through the index theorems of Hens (1991) for GEI models with a single commodity, Kubler and Schmedders (1997) for stochastic finance economies, Schmedders (1999) for an open (but not necessarily generic) set of two-period economies using a homotopy algorithm, Momi (2003) and Predtetchinski (2006) for economies where the degree of incompleteness is even, and Anderson and Raimondo (2007) for GEI models with no Hart points.

The goal of this paper is to construct an index theorem for smooth infinite economies.[2] This will show that the number of equilibria of smooth infinite economies is odd and hence it provides an alternative proof of existence of equilibria. Additionally, it provides conditions that guarantee global uniqueness of equilibria.[3] In order to do so, we will use an infinite-dimensional analogue of the Poincaré-Hopf Theorem that was proposed by Tromba (1978). Tromba's result is valid only for vector fields that have a very particular structure — that of a zero-Rothe (Z-Rothe or ZR) vector field — a notion that we introduce in this paper to the economic literature. A substantial part of this paper goes into showing that aggregate excess demand functions of smooth infinite economies do indeed define a Z-Rothe vector field.

This paper is structured as follows. In order to fix ideas, we begin in section 2 by reviewing a recent example of an infinite economy with complete financial markets studied by Crès et al. (2009) as a framework to understand jumps (or the lack thereof) in asset prices. We then set the market and define aggregate excess demand functions in our setting; as usual, we will interpret

[2] Roughly speaking, smooth infinite economies are models of competitive markets where consumption bundles, and endowments, are continuous functions over the parameters that define them.

[3] In infinite dimensions, one of the few results on global uniqueness has been provided by Dana (1993) by taking into consideration a model of a pure exchange economy where the agents' consumption space is $L^p_+(\mu)$ and agents have additively separable utilities which fulfill the (RA) assumption, that is, that the agents' relative risk aversion coefficients are smaller than one. In this case, Dana shows that one can work with the space of utility weights to avoid using the demand approach that may not be well defined. Dana finally shows that if utilities fulfill the (RA) assumption then the excess utility map is gross substitute which in turn implies existence and global uniqueness.

them as vector fields on the space of prices.

Sections 3 and 4 study in detail the structure of aggregate excess demand functions of smooth infinite economies. First, subsection 3.1 contains a quick review of the basic definitions of Fredholm theory, which are mathematical tools needed to extend differential topology to infinite dimensions (further mathematical definitions are included in an appendix). In section 3.2 we review the determinacy results obtained previously in Covarrubias (2010) showing that most excess demand functions have isolated zeros; that is, that equilibria are locally unique. This guarantees that it makes sense to actually count the number of equilibria. As we mentioned above, this result is by no means the strongest determinacy result available: that would be Shannon and Zame (2002). However, the fact that the determinacy result is written in terms of aggregate excess demand functions makes the exposition of this paper more fluid.

In section 4, we review the notion of Z-Rothe vector fields as developed by Tromba (1978). When an aggregate excess demand function is Z-Rothe, we can define a suitable index of equilibrium prices, that is, an index of zeros of a vector field. As mentioned previously, the Poincaré-Hopf Theorem in infinite dimensions holds for Z-Rothe vector fields. Our Theorem 3 below will show that indeed smooth infinite economies have an aggregate excess demand function that is Z-Rothe. Finally, in section 5, we prove the main theorem where we construct an index theorem for smooth infinite economies. We show that the sum of indices of equilibrium prices is constant and equal to 1. We give a corollary to the index theorem analogous to (Dierker, 1972), giving a new proof of existence of equilibrium and analyzing what condition an excess demand function needs to fulfill to give rise to a globally unique equilibrium.

2 The Market

In order to fix ideas, we begin by explaining a recent examples studied by Crès et al. (2009) of a smooth infinite economy with financial markets. Many more examples of infinite economies may be found, for instance, in Mas-Colell (1991), Mas-Colell and Zame (1991) and Chichilnisky and Zhou (1998).[4] Throughout, we will denote

$$\mathbb{R}^n_{++} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_j > 0, \forall j = 1, \ldots, n\}.$$
$$\mathbb{R}^n_{+} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \geq 0, \forall j = 1, \ldots, n\}.$$

2.1 An example of a smooth infinite economy

In this example of a pure exchange economy with complete financial markets, there are two time periods ($t = 0, 1$). Uncertainty at $t = 1$ is represented by a set of states $M = [0, 1]$ and the density of these states is given by a C^1 map $\pi : M \to \mathbb{R}_+$. There are $i = 1, \ldots, I$ agents and n commodities at each time period and at each state. A consumption bundle is hence a pair $x_i = (x_i^0, x_i^1)$ where at $t = 0$ consumption is a vector $x_i^0 \in \mathbb{R}^n_+$ and at $t = 1$ it is a map $x_i^1 : M \to \mathbb{R}^n_+$. Each of the I agents is equipped with a $t = 0$ initial endowment $\omega_i^0 \in \mathbb{R}^n_{++}$ and a C^1 endowment at $t = 1$ of the form $\omega_i^1 : M \to \mathbb{R}^n_{++}$. Preferences of each agent i are represented by a utility of the form

$$W^i(x_i) = \tilde{u}^i(x_i^0) + \int_M \tilde{u}^i(x_i^1(s))\, \pi(s)\, ds.$$

Crès et al. (2009) (but also see Mas-Colell (1991) and Chichilnisky and Zhou (1998)) show that if

$$(p, x_1, \ldots, x_I) = \left((p^0, p^1), (x_1^0, x_1^1), \ldots, (x_I^0, x_I^1)\right)$$

is an equilibrium, then p^1 and $\{x_i^1\}_{i=1}^I$ are all continuous maps from $M = [0, 1]$ to $\mathbb{R}^n_{++}$. That is, initial endowments, consumption and prices are all elements of the same space of continuous maps from M to $\mathbb{R}^n_{++}$.[5]

[4] Other examples of continuous economies usually arise from considerations of time varying in $[0, T]$ or when commodities are parametrized by their characteristics.

[5] Another way of interpreting this result is that, when markets are complete, there are no "jumps" in equilibrium prices when interpreted as functions of the parameter space M.

2.2 Other examples of smooth infinite economies

Other examples of smooth infinite economies arise from considerations of continuous time. For instance, suppose that in an economy the consumption of n goods is done continuously through time $t \in [0, T]$. Then, a continuous function $x^i : [0, T] \to \mathbb{R}^n_{++}$ represents the consumption of the n goods by agent i at time t. Alternatively, $x(t)$ may represent a continuous instantaneous rate of consumption.

2.3 The economy

In this section we set the market. We will define the consumption and price sets, preferences and demand functions. We remind the reader that at the end of the paper is an appendix with some mathematical definitions.

2.3.1 The commodity, consumption and price sets

Let M be a compact (i.e. closed and bounded) subset of $\mathbb{R}^m$ for some m. The example to keep in mind is $M = [0, 1]$ or $M = [0, 1]^T$ if we allow for several time periods. The **commodity space** is the set $C(M, \mathbb{R}^n)$ of all continuous functions from M to $\mathbb{R}^n$ equipped with the norm

$$\|f\| = \sup_{t \in M} \|f(t)\|_{\mathbb{R}^n}$$

with the standard norm $\| \cdot \|_{\mathbb{R}^n}$ on $\mathbb{R}^n$. Abusing notation, we will sometimes drop the explicit mention of $\mathbb{R}^n$.

The **consumption set** is then $X = C^{++}(M, \mathbb{R}^n)$, the positive cone of $C(M, \mathbb{R}^n)$. This set consists of all the continuous functions from M to $\mathbb{R}^n$ of the form $f = (f_1(t), \ldots, f_n(t))$ such that $f_j(t) > 0$ for all $j = 1, \ldots, n$ and for all $t \in M$.

Strictly speaking, prices should be in the positive cone of the dual space of the commodity space $C(M, \mathbb{R}^n)$. However, it is shown in Mas-Colell (1991), Chichilnisky and Zhou (1998) and Crès et al. (2009) that with separable

However, assuming incomplete markets, Crès et al. (2009) provide a robust example of an economy with discontinuities.

utilities only a small subset of this space can support equilibria and we can actually then consider the **price space** to simply be

$$S = \left\{ P \in C^{++}(M, \mathbb{R}^n) : \|P\| = 1 \right\}$$

where

$$\|P\| = \sup_{t \in M} \|P(t)\|_{\mathbb{R}^n}$$

with the standard metric $\| \cdot \|_{\mathbb{R}^n}$ on $\mathbb{R}^n$. We also denote by $\langle \cdot, \cdot \rangle$ the inner product on $C(M, \mathbb{R}^n)$ so that if $f, g \in C(M, \mathbb{R}^n)$ then

$$\langle f, g \rangle = \int_M \langle f(t), g(t) \rangle_{\mathbb{R}^n} \, dt$$

with the standard inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ in $\mathbb{R}^n$. Again, abusing notation, we will sometimes drop the explicit mention of $\mathbb{R}^n$ in the inner product.

2.3.2 Preferences and individual demand functions

We consider a finite number I of agents. Each agent is equipped with **preferences represented by a utility function** of the form

$$U^i(x) = \int_M u^i(x(t), t) dt$$

where $u^i(x(t), t) : \mathbb{R}^n_{++} \times M \to \mathbb{R}$ is a strictly monotonic, concave, C^2 function where $\{y \in R^n_{++} : u^i(y, t) \geq u^i(x, t)\}$ is closed. Hervés-Beloso and Monteiro (2009) show that such representation is possible and Chichilnisky and Zhou (1998) show that these assumptions on $u^i(x(t), t)$ imply that $U^i(x)$ is strictly monotonic, concave and twice Fréchet differentiable.

The **individual demand functions** $f_i : S \times (0, \infty) \to X$ of each agent i are solutions to the optimization problem

$$f_i(P, y) = arg \left[\max_{\langle P(t), x \rangle = y} U^i(x) \right].$$

Since we will fix preferences, an **exchange economy** will be parameterized by the initial endowments $\omega_i \in X$ of each agent $i = 1, \ldots, I$. Finally, denote $\omega = (\omega_1, \ldots, \omega_I) \in \Omega = X^I$.

2.3.3 Aggregate excess demand function.

For a fixed economy $\omega \in \Omega$ the **aggregate excess demand function** is a map $Z_\omega : S \to C(M, \mathbb{R}^n)$ defined by

$$Z_\omega(P) = \sum_{i=1}^{I} \left(f_i\left(P, \langle P, \omega_i \rangle\right) - \omega_i\right).$$

We also define $Z : \Omega \times S \to C(M, \mathbb{R}^n)$ by the evaluation

$$Z(\omega, P) = Z_\omega(P).$$

It is shown in Covarrubias (2010) that it satisfies $\langle P, Z_\omega(P) \rangle = 0$ for all $P \in S$, which in turn implies that Z_ω can be interpreted as a vector field on the set of prices S. When thought of as a vector field, we will write $Z_\omega : S \to TS$ where TS is the tangent bundle of S.

Definition 1. *We say that $P \in S$ is an* ***equilibrium*** *of the economy $\omega \in \Omega$ if $Z_\omega(P) = 0$. We denote the* ***equilibrium set*** *by*

$$\Gamma = \{(\omega, P) \in \Omega \times S : Z(\omega, P) = 0\}.$$

3 Determinacy of Equilibria

We wish to explore the structure of aggregate excess demand functions and since we will be using tools of differential topology in infinite dimensions, we would like our maps to be Fredholm as introduced by Smale (1965). We briefly remind the reader of these definitions.[6]

3.1 Fredholm index theory

A linear **Fredholm operator** is a continuous linear map $L : E_1 \to E_2$ from one Banach space to another with the properties:

1. dim ker $L < \infty$;
2. range L is closed;
3. coker $L := E_2/\text{range}L$ has finite dimension.

If L is a Fredholm operator, then its **index** is dim ker$L -$ dim coker$L$, so that the index of L is an integer.

[6]As a motivation for Fredholm maps, suppose that we consider a linear map T between any two vector spaces V and W. We may ask ourselves, what conditions would T need to satisfy in order for it to be a bijection, that is, a map that is both injective and surjective? If T were a bijection, this would also mean that T is invertible.

There are two basic results of linear algebra that would answer this question. First, recall that the kernel of T, or $\ker T$, consists of those points of V that are mapped into zero in W under T. In order for T to be injective, we would require that $\ker T = \{0\}$. Similarly, recall that the range of T, or range T, consists of all those points that are in the image under T in W. For T to be surjective, we would require that range $T = W$.

As it happens, these two conditions are rather restrictive. Fredholm operators were introduced since, loosely speaking, they are "almost invertible": they are "almost injective" and "almost surjective". By this we mean that $\ker T$ is a finite-dimensional subspace of V (not just the point $\{0\}$ but also not an infinite-dimensional set) and the range of T "misses" the entire set W only by a finite-dimensional subspace.

Expanding these notions, two linear maps $T : V \to W$ and $S : W \to V$ are "pseudo-inverses" to each other if $ST = I + G_1$ and $TS = I + G_2$, where I is the identity and G_1 and G_2 are two maps with finite-dimensional range. In other words, while ST and TS are not the identity, they fail to be so only by a "compact perturbation" of the identity. It can be shown that $T : V \to W$ will have a pseudo-inverse if and only if T is a Fredholm operator. Fredholm maps are the nonlinear notion of a Fredholm operator.

A **Fredholm map** is a C' map $f : M \to V$ between differentiable manifolds locally like Banach spaces such that for each $x \in M$ the derivative $Df(x) : T_x M \to T_{f(x)} V$ is a Fredholm operator. The **index** of f is defined to be the index of $Df(x)$ for some x. If M is connected, this definition does not depend on x.

3.2 Determinacy of equilibria

In our previous work (2010) we have shown that the excess demand function $Z_\omega : S \to C(M, \mathbb{R}^n)$ of economy $\omega \in \Omega$ is a Fredholm map of index zero. Since we would like to count the number of price equilibria of an economy, the first result that we need to establish is that generically equilibria will be isolated. Below we remind the reader the notion of a regular economy and of a regular price system.

Definition 2. *We say that an economy is* ***regular*** *(resp.* ***critical****) if and only if ω is a regular (resp. critical) value of the projection $\pi : \Omega \times S|_\Gamma \to \Omega$.*

Definition 3. *Let Z_ω be the excess demand of economy ω. A price system $P \in S$ is a* ***regular equilibrium price*** *system if and only if $Z_\omega(P) = 0$ and the derivative of $Z_\omega(P)$, denote $DZ_\omega(P)$, is surjective.*

In our previous work (2010) we showed the relation between regular economies and regular equilibrium prices.

Theorem 1. *(Covarrubias, 2010) The economy $\omega \in \Omega$ is regular if and only if all equilibrium prices of Z_ω are regular.*

Theorem 2 showed that for most economies, its aggregate excess demand function will have isolated zeros. Hence, it makes sense to try to count them.

Theorem 2. *(Covarrubias, 2010) Almost all economies are regular. That is, the set of economies $\omega \in \Omega$ that give rise to an excess demand function*

Z_ω with only regular equilibrium prices, are residual in Ω.

Since we haven shown that *most* excess demand functions Z_ω will have isolated zeros, we will drop the explicit dependence on a generic economy ω and will simply write Z. Again, we remind the reader that Theorem 2 above is not the strongest determinacy result available (*cf.* Shannon and Zame (2002)).

4 Z-Rothe Vector Fields

Knowing that the aggregate excess demand function is a vector field on the price space, and that is a Fredholm map for which we know its index, we would like to know if it has the structure of a Z-Rothe vector field as developed by Tromba (1978).[7]

To define this, let E be any Banach space and $\mathcal{L}(E)$ be the set of linear continuous maps from E to itself. Denote by $G\mathcal{L}(E)$ the general linear group of E; that is, the set of invertible linear maps in $\mathcal{L}(E)$. Let $C(E)$ be the linear space of compact linear maps from E to itself. We write

$$\mathcal{L}_C(E) = \{T : T = I + C, I \text{ the identity}, C \in C(E)\}.$$

We write $\mathcal{S}(E) \subset G\mathcal{L}(E)$ to denote the maximal starred neighborhood of the identity in $G\mathcal{L}(E)$. Formally,

$$\mathcal{S}(E) = \{T \in G\mathcal{L}(E) : (\alpha T + (1-\alpha)I) \in G\mathcal{L}(E), \forall \alpha \in [0,1]\}.$$

The **Rothe set** of E is defined as

$$\mathcal{R}(E) = \{A : A = T + C, T \in \mathcal{S}(E), C \in C(E)\}$$

and its invertible members by $G\mathcal{R}(E) = \mathcal{R}(E) \cap G\mathcal{L}(E)$.

[7] We remind the reader that an appendix is provided at the end of the paper with some relevant mathematical concepts.

Definition 4. *A C^1 vector field v on a Banach manifold N is a* ***Z-Rothe vector field*** *if whenever $v(P) = 0$, the Frèchet derivative $Dv(P) \in \mathcal{R}(T_P N)$.*

Theorem 3. *The excess demand function, Z is a Z-Rothe vector field.*

Proof. To show that $Z : S \to TS$ is Z-Rothe, where TS denotes the tangent bundle to S, we need to show that whenever $Z(P) = 0$, then its Fréchet derivative DZ can be written of the form $T + C$ where T is an invertible map and in the maximal starred neighborhood of the identity in $G\mathcal{L}(T_P S)$ and where C is a compact map. While the explicit calculation of Z has been done in previous work (2010), we nevertheless put to use part of those calculations in this proof for the sake of completeness.

Recall that the consumers' problem is given by

$$\max_{x_i \in X} U^i(x_i) \quad \text{s.t.} \quad \langle P, x_i \rangle = y_i$$

where

- $X = C^{++}(M, \mathbb{R}^n)$;
- $U^i : X \to \mathbb{R}$ is given by $U^i(x_i) = \int_M u^i(x_i(t), t)\, dt$;
- $u^i : \mathbb{R}^n_{++} \times M \to \mathbb{R}$ with the usual assumptions of smoothness, strict concavity and monotonicity;
- In principle, P is an element of the positive cone of the dual of $C(M, \mathbb{R}^n)$. However, we have explained that with separable utilities, actually P is an element of $C^{++}(M, \mathbb{R}^n)$;
- Furthermore, we normalise so that

$$P \in S = \left\{ P \in C^{++}(M, \mathbb{R}^n) : \|P\| = 1 \right\};$$

- $y_i \in (0, \infty)$.

Notice that $P \in S$ is an independent (exogenous) variable of the consumer problem. Also, eventually, we will make $y_i = \langle P, \omega_i \rangle$.

Now, because of the assumptions that we have placed on the utility functions u^i (smoothness, concavity, monotonicity), this implies that for each $P \in S$ and for each $y_i \in (0, \infty)$ the optimization problem has a unique solution that we will denote by $f_i(P, y_i)$ where $f_i : S \times (0, \infty) \to X$.

The first order optimality conditions can then be written as:

$$y_i = \langle P, f_i(P, y_i) \rangle \tag{1}$$
$$DU^i(f_i(P, y_i)) = \lambda_i(P, y_i) \cdot P \tag{2}$$

where DU^i denotes the Fréchet derivative of $U^i : X \to \mathbb{R}$ and $\lambda_i : S \times (0, \infty) \to \mathbb{R}$ is a Lagrange multiplier.

The strategy is to calculate the total derivatives of equations (1) and (2) and solve for $Df_i(P, y_i)$. We will exploit the simplicity of $U^i(x)$ written in terms of u^i. Hence, we first write equations (1) and (2) as

$$y_i = \langle P, f_i(P, y_i) \rangle \tag{3}$$
$$u^i_x(f_i(P, y_i), t) = \lambda_i(P, y_i) \cdot P \tag{4}$$

where u^i_x denotes the partial derivative of u^i with respect to x. Taking total derivatives on both sides of equations (3) and (4) we get

$$Dy_i = f_i(P, y_i) + \langle P, Df_i(P, y_i) \rangle$$
$$u^i_{xx}(f_i(P, y_i), t) \cdot Df_i(P, y_i) = \lambda_i(P, y_i) + P \cdot D\lambda_i(P, y_i)$$

where we write $\langle P, Df_i(P, w)_i \rangle$ to denote the linear transformation Df_i composed with the linear transformation P.

Simplifying, and remembering that since $u^i(x)$ is concave, the linear transformation (u^i_{xx}) is negative definite and hence (u^i_{xx}) is invertible for each t, we now have

$$Dy_i = f_i(P, y_i) + \langle P, Df_i(P, y_i) \rangle \tag{5}$$
$$Df_i(P, y_i) = \lambda_i(P, y_i) \, (u^i_{xx})^{-1} + (u^i_{xx})^{-1} \, P \cdot D\lambda_i(P, y_i). \tag{6}$$

Making a substitution of the expression of Df_i found in (6) into Dy_i of equation (5), and remembering that P is linear, we get

$$\begin{aligned} Dy_i &= f_i(P, y_i) + \langle P, Df_i(P, y_i)\rangle \\ &= f_i(P, y_i) + \langle P, \lambda_i(P, y_i)\,(u^i_{xx})^{-1} + D\lambda_i(P, y_i)\,(u^i_{xx})^{-1}\,P\rangle \\ &= f_i(P, y_i) + \langle P, \lambda_i(P, y_i)\,(u^i_{xx})^{-1}\rangle + \langle P, D\lambda_i(P, y_i)\,(u^i_{xx})^{-1}\,P\rangle \\ &= f_i(P, y_i) + \lambda_i(P, y_i)\,(u^i_{xx})^{-1}P + D\lambda_i(P, y_i)\,\langle P, (u^i_{xx})^{-1}\,P\rangle \end{aligned}$$

Therefore,

$$D\lambda_i(P, y_i) = \frac{1}{\langle P, (u^i_{xx})^{-1}\,P\rangle}\left[Dy_i - f_i(P, y_i) - \lambda_i(P, y_i)\,(u^i_{xx})^{-1}P\right] \tag{7}$$

where the denominator $\langle P, (u^i_{xx})^{-1}\,P\rangle$ does not vanish since P and $(u^i_{xx})^{-1}$ are positive operators.

We substitute the expression of $D\lambda_i$ found in (7) into (6) to get,

$$\begin{aligned} Df_i(P, y_i) &= \lambda_i(P, y_i)\,(u^i_{xx})^{-1} + D\lambda_i(P, y_i)\,(u^i_{xx})^{-1}\,P \\ &= \lambda_i(P, y_i)\,(u^i_{xx})^{-1} + \\ &+ \frac{(u^i_{xx})^{-1}\,P}{\langle P, (u^i_{xx})^{-1}\,P\rangle}\left[Dy_i - f_i(P, y_i) - \lambda_i(P, y_i)\,(u^i_{xx})^{-1}P\right] \end{aligned}$$

What we have shown is that $Df_i(P, w)$ can be written as the sum of the invertible operator

$$\lambda_i(P, y_i)\,(u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\,P}{\langle P, (u^i_{xx})^{-1}\,P\rangle}Dy_i$$

and the finite rank operator

$$-\frac{(u^i_{xx})^{-1}\,P}{\langle P, (u^i_{xx})^{-1}\,P\rangle}\left[f_i(P, y_i) + \lambda_i(P, y_i)(u^i_{xx})^{-1}\,P\right]$$

Now, let $y_i = \langle P, \omega_i \rangle$ and recall that $Z : S \to C(M, \mathbb{R}^n)$ is given by

$$Z(P) = \sum_{i=1}^{I} \left(f_i(P, \langle P, \omega_i \rangle) - \omega_i \right)$$

and so its Fréchet derivative $DZ : TS \to TC(M, \mathbb{R}^n)$ is given by

$$\begin{aligned} DZ(P) &= \sum_{i=1}^{I} Df_i(P, y_i) \\ &= \sum_{i=1}^{I} \left\{ \lambda_i(P, y_i)\, (u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\, P}{\langle P, (u^i_{xx})^{-1}\, P \rangle} Dy_i \right\} + \\ &\quad + \sum_{i=1}^{I} \left\{ -\frac{(u^i_{xx})^{-1}\, P}{\langle P, (u^i_{xx})^{-1}\, P \rangle} \left[f_i(P, y_i) + \lambda_i(P, y_i)(u^i_{xx})^{-1}\, P \right] \right\} \end{aligned}$$

Finally, noticing again that since $u^i(x)$ is concave, the linear transformation (u^i_{xx}) is negative definite and hence (u^i_{xx}) is invertible. Additionally, the sum of negative-definite linear transformations is again negative-definite. Hence,

$$\sum_{i=1}^{I} \left\{ -\frac{(u^i_{xx})^{-1}\, P}{\langle P, (u^i_{xx})^{-1}\, P \rangle} \left[f_i(P, y_i) + \lambda_i(P, y_i)(u^i_{xx})^{-1}\, P \right] \right\}$$

has finite rank, and

$$\sum_{i=1}^{I} \left\{ \lambda_i(P, y_i)\, (u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\, P}{\langle P, (u^i_{xx})^{-1}\, P \rangle} Dy_i \right\}$$

is invertible. Therefore, DZ is written as the sum of an invertible operator and an operator of finite rank. All we need to show then is that

$$\alpha \left[\sum_{i=1}^{I} \left\{ \lambda_i(P, y_i)\, (u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\, P}{\langle P, (u^i_{xx})^{-1}\, P \rangle} Dy_i \right\} \right] + (1 - \alpha) I$$

is invertible for all $\alpha \in [0, 1]$. But this sum is just a homotopy of positive-definite operators. □

5 The Index Theorem of Smooth Infinite Economies

Knowing now that most economies are regular, we need to find a right way of counting the number of equilibria. With an excess demand function that is a Z-Rothe vector field, we may use tools from infinite-dimensional differential topology that resemble the finite-dimensional case. In particular, we will review the notion of the Euler characteristic proposed by Tromba (1978) and, with its aid, construct an index theorem for smooth infinite economies.

5.1 The Euler characteristic of vector fields

A zero P of a vector field v is **nondegenerate** if $Dv(P) : T_P N \to T_P N$ is an isomorphism. Now, suppose that a Z-Rothe vector field v has only nondegenerate zeros, and let P be one of them. Then, $Dv(P) \in G\mathcal{R}(T_P N)$. Tromba (1978) shows that $G\mathcal{R}(T_P N)$ has two components; $G\mathcal{R}^+(E)$ denotes the component of the identity. Define

$$sgn\, Dv(P) = \begin{cases} +1, & \text{if } Dv(P) \in G\mathcal{R}^+(T_P N) \\ -1, & \text{if } Dv(P) \in G\mathcal{R}^-(T_P N) \end{cases}.$$

The **Euler characteristic** is then given by the formula

$$\chi(v) = \sum_{P \in Zeros(v)} sgn Dv(P)$$

Tromba also shows that this Euler characteristic is invariant under homotopy of vector fields. All we have to do is to construct a vector field on S that has only one singularity and that is homotopic to the aggregate excess demand Z.

5.2 The index theorem of smooth infinite economies

We are finally ready to construct an index theorem for smooth infinite economies. Given that Fredholm maps could be simply described as maps between infinite-dimensional spaces that most closely resemble the finite-dimensional case, it is no surprise that the proof of the index theorem in our setting is very similar to that of Dierker (1972) or Mas-Colell (1985).

Suppose that the excess demand satisfies the desirability assumption of Dierker (1972),[8] namely that if $P_n \in S$ and $P_n \to P \in \partial S$ (the boundary of S), then

$$\|Z(P_n)\| \to \infty.$$

Suppose also that Z is bounded below and that there are only finitely many zeros. The final ingredient before proving the index theorem is to check that the excess demand function is a vector field that is outward pointing along the boundary of S. To see this, consider a sequence of prices $P_n \to P \in \partial S$. Since we have assumed that Z is bounded from below and that $\|Z(P_n)\| \to \infty$, then the limit of $[1/\|Z(P_n)\|]\, Z(P_n)$ must converge to a point $z \in C^{++}(M, \mathbb{R}^n)$. Hence, Z is inward-pointing and therefore $-Z$ is outward-pointing along ∂S.

Theorem 4. *Suppose that an aggregate excess demand function Z is bounded from below and that it satisfies the boundary assumption. Suppose also that Z has only finitely many singularities and that they are all nondegenerate. Then,*

$$\sum_{P \in Zeros Z} sgn\,[-DZ(P)] = 1.$$

Proof. The proof follows closely the proof of the index theorem in finite dimensions (see Dierker (1982) and Mas-Colell (1985)): it consists of two steps. The first consists in constructing a specific vector field on S, which we call Z^Q, that has only one zero, it is inward-pointing along the boundary of S, and for which calculating the index is simple. The second step consists in showing that the excess demand function Z is properly homotopic to the vector field Z^Q and that this proper homotopy is through Z-Rothe vector

[8]This assumption, expresses the idea that every commodity is desired by at least one agent.

fields.

Let $\bar{S}$ denote the closure of S. For any fixed $Q \in C^{++}(M, \mathbb{R}^n)$ define the vector field $Z^Q : \bar{S} \to TS$ given by

$$Z^Q(P) = \left[\frac{Q}{\langle P, Q\rangle}\right] - P.$$

By construction, $Z^Q(P)$ has only one zero and is inward-pointing on the boundary. Its derivative $DZ^Q_{(P)} : T\bar{S} \to T(TS)$ is given by

$$DZ^Q_{(P)}(h) = -\frac{Q\langle h, Q\rangle}{\langle P, Q\rangle^2} - h$$

where

$$h \mapsto -\frac{Q\langle h, Q\rangle}{\langle P, Q\rangle^2}$$

is compact and

$$h \mapsto -h$$

is invertible; then $DZ^Q \in \mathcal{R}(T_P S)$.

Now let

$$-\frac{Q\langle h, Q\rangle}{\langle P, Q\rangle^2} - h = h'. \tag{8}$$

We need to solve for h. Then,

$$Q\langle h, Q\rangle + h\langle P, Q\rangle^2 = -h'\langle P, Q\rangle^2.$$

Acting Q on both sides we get,

$$\langle Q, Q\rangle\langle h, Q\rangle + \langle h, Q\rangle\langle P, Q\rangle^2 = -\langle h', Q\rangle\langle P, Q\rangle^2.$$

Solving for $\langle h, Q\rangle$ we get

$$\langle h, Q\rangle = \frac{-\langle h', Q\rangle\langle P, Q\rangle^2}{\langle Q, Q\rangle + \langle P, Q\rangle^2}$$

where the denominator never vanishes since $Q \in C^{++}(M, \mathbb{R}^n)$. Substituting $\langle h, Q\rangle$ in (8) we then get

$$h = h' + \frac{Q}{\langle P, Q\rangle^2}\left[\frac{\langle h', Q\rangle\langle P, Q\rangle^2}{\langle Q, Q\rangle + \langle P, Q\rangle^2}\right].$$

This shows that DZ^Q is invertible and therefore $DZ^Q \in G\mathcal{R}(T_P S)$. Furthermore, since it is not in the same component of the identity it has to be in $G\mathcal{R}^-(T_P S)$ and its only zero has index -1. The vector field Z^Q is inward pointing so reversing orientation will make it outward pointing with index of +1.

Up to this stage we have constructed a specific vector field on S, which we called Z^Q, that has only one zero, it is inward-pointing along the boundary of S, and whose index we have shown to be +1. All that we need to do is to show that the excess demand function Z is properly homotopic to the vector field Z^Q and that this proper homotopy is through Z-Rothe vector fields.

Consider then the homotopy $F : S \times [0, 1] \to C(M, \mathbb{R}^n)$ given by

$$F(P, \alpha) = \alpha Z(P) + (1 - \alpha)Z^Q(P)$$

We have seen that

$$DZ(P) = \sum_{i=1}^{I}\left\{\lambda_i(P, y_i)\,(u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\,P}{\langle P, (u^i_{xx})^{-1}\,P\rangle}Dy_i\right\} + \\ + \sum_{i=1}^{I}\left\{-\frac{(u^i_{xx})^{-1}\,P}{\langle P, (u^i_{xx})^{-1}\,P\rangle}\left[f_i(P, y_i) + \lambda_i(P, y_i)(u^i_{xx})^{-1}\,P\right]\right\}$$

and

$$DZ^Q(P) = -\frac{Q\langle \cdot, Q\rangle}{\langle P, Q\rangle^2} - I$$

Hence

$$\begin{aligned}
DF(P,\alpha) &= \alpha DZ(P) + (1-\alpha) DZ^Q(P) \\
&= \alpha \sum_{i=1}^{I} \left\{ \lambda_i(P, y_i)\, (u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\, P}{\langle P, (u^i_{xx})^{-1}\, P\rangle} D y_i \right\} + (1-\alpha)\left\{-I\right\} \\
&+ \alpha \sum_{i=1}^{I} \left\{ -\frac{(u^i_{xx})^{-1}\, P}{\langle P, (u^i_{xx})^{-1}\, P\rangle} \left[f_i(P, y_i) + \lambda_i(P, y_i) (u^i_{xx})^{-1}\, P \right] \right\} \\
&+ (1-\alpha)\left\{ -\frac{Q\langle \cdot, Q\rangle}{\langle P, Q\rangle^2} \right\}
\end{aligned}$$

Finally notice that since $\alpha > 0$ and $1 - \alpha > 0$, then

$$\alpha \sum_{i=1}^{I} \left\{ \lambda_i(P, y_i)\, (u^i_{xx})^{-1} + \frac{(u^i_{xx})^{-1}\, P}{\langle P, (u^i_{xx})^{-1}\, P\rangle} D y_i \right\} + (1-\alpha)\left\{-I\right\}$$

is invertible, and

$$\alpha \sum_{i=1}^{I} \left\{ -\frac{(u^i_{xx})^{-1}\, P}{\langle P, (u^i_{xx})^{-1}\, P\rangle} \left[f_i(P, y_i) + \lambda_i(P, y_i) (u^i_{xx})^{-1}\, P \right] \right\} + (1-\alpha)\left\{ -\frac{Q\langle \cdot, Q\rangle}{\langle P, Q\rangle^2} \right\}$$

has finite rank.

Hence, Z is properly homotopic to the vector field Z^Q and that this proper homotopy is through Z-Rothe vector fields.

□

6 Concluding Remarks

We conclude from Theorem 4 that the number of equilibria of smooth infinite economies generically is odd. In particular, it cannot be zero so this gives a new proof of existence.

Also, as a corollary of Theorem 4, we can provide an infinite-dimensional analogue of Dierker (1972); Dierker shows the following.

Theorem 5. *(Dierker, 1972) If the Jacobian of the excess supply function is positive at all Walras equilibria, then there is exactly one equilibrium.*

We have shown that:

Corollary 1. *If the sign of the derivative of the excess supply function is positive at all Walras equilibria, i.e., if $-DZ(P) \in G\mathcal{R}^+(T_P S)$, then there is exactly one equilibrium.*

The logic behind this corollary is simple. The sign of the derivative of the excess supply function is either $+1$ or -1. If this sign is positive at all Walras equilibria, and the sum of these needs to be equal to $+1$, there can only be one.

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Appendix: Mathematical Definitions.

Definition 5. *A **homotopy** is any family of maps* $f_t : X \to Y$*, between topological spaces,* $t \in I = [0,1]$*, such that the associated map* $F : X \times I \to Y$ *given by* $F(x,t) = f_t(x)$ *is continuous. One says that two maps* $f_0, f_1 : X \to Y$ *are **homotopic** if there exists a homotopy* f_t *connecting them.*

Definition 6. *A topological space is said to be **contractible** if the identity map* $i_X : X \to X$ *is homotopic to a constant map.*

Definition 7. *A **Banach space*** $(X, \|\cdot\|)$ *is a normed vector space (over the real numbers throughout) that is complete with respect to the metric* $d(x,y) = \|x - y\|$.

Definition 8. *A **Hilbert space*** H *is a vector space with a positive-definite inner product* $\langle \cdot, \cdot \rangle$ *that defines a Banach space upon setting* $\|x\|^2 = \langle x, x \rangle$ *for* $x \in H$.

Definition 9. *A **bounded linear functional*** $h(x)$ *defined on a Banach space* X *is a linear mapping* $X \to \mathbb{R}$ *such that* $|h(x)| \leq K\|x\|_X$ *for some constant* K *independent of* $x \in X$*. The set of all bounded linear functionals on* X*, denoted* X^**, is called the **conjugate space** of* X*. It is a Banach space with respect to the norm* $\|h\| = sup|h(x)|$ *over the sphere* $\|x\|_X = 1$*. If* $(X^*)^* = X$*, then the space* X *is called **reflexive**.*

Definition 10. *One says that a set* M *of a Banach space* X *is **compact set** if* M *is closed (in the norm topology) and such that every sequence in* M *contains a strongly convergent subsequence.*

Definition 11. *A linear operator* L *with domain* X *and range contained in* Y*, (*X*,*Y *Banach spaces) is a **bounded linear operator** if there is a constant* K *independent of* $x \in X$ *such that* $\|Lx\|_Y \leq K\|x\|_X$ *for all* $x \in X$*. The set of such maps for fixed* X*,*Y *is again a Banach space, denoted* $L(X,Y)$ *with respect to the norm* $\|L\| = sup\|Lx\|_Y$ *for* $\|x\|_X = 1$.

Definition 12. *A linear operator* $C \in L(X,Y)$ *is called a **compact operator** if for any bounded set* $B \subset X$*,* $C(B)$ *is conditionally compact in* Y*. Bounded linear mappings with finite-dimensional ranges are automatically compact; and conversely, if* X *and* Y *are Hilbert spaces, then a compact linear mapping* C *is the uniform limit of such mappings.*

Relevant properties of linear compact operators. Let $C \in L(X, X)$ be compact, and set $L = I + C$. Then

1. L has closed range.
2. dim ker$L$ = dim coker$L < \infty$.
3. there is a finite integer β such that $X = \ker(L^\beta) \oplus \text{range}(L^\beta)$ and L is a linear homeomorphism of range(L^β) onto itself.

Definition 13. *Let $f \in C^1(U, Y)$, $U \subset X$, X, Y Banch spaces. Then, $x \in U$ is a* ***regular point*** *for f if $f'(x)$ is a surjective linear mapping in $L(X, Y)$. If $x \in U$ is not regular, x is called* ***singular point****. Similarly,* ***singular values*** *and* ***regular values*** *y of f are defined by considering the sets $f^{-1}(y)$. If $f^{-1}(y)$ has a singular point, y is called a singular value, otherwise y is a regular value.*

Definition 14. *An operator $f \in C^0(X, Y)$ is said to be a* ***proper operator*** *if the inverse image of any compact set C in Y, $f^{-1}(C)$ is compact in X. The importance of this notion resides in the fact that the properness of an operator f restricts the size of the solution set $S_p = \{x : x \in X, f(x) = p\}$ for any fixed $p \in Y$.*

Definition 15. *A map f betwen topological space X, Y is said to be a* ***proper map*** *if the inverse image of each compact subset of Y is a compact subset of X.*