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N° 2018-05

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Acceptance: Acyclicity and Dropping Strategies**

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April 2018

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# Stability of Equilibrium Outcomes under Deferred Acceptance: Acyclicity and Dropping Strategies\*

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**Abstract:** We consider the problem of matching a set of medical students to a set of medical residency positions (hospitals) under the assumption that hospitals' preferences over groups of students are responsive. In this context, we study the preference revelation game induced by the student proposing deferred acceptance mechanism. We show that the acyclicity of the hospitals' preference profile (Romero-Medina and Triossi, 2013a) is a necessary and sufficient condition to ensure that the outcome of every Nash equilibrium in which each hospital plays a dropping strategy is stable.

**Keywords:** matching, stability, acyclicity, dropping strategies, Nash equilibria

**JEL Classification:** C78, D47

**Resumen:** Se considera el problema de asignar médicos residentes (estudiantes) a residencias médicas (hospitales) bajo el supuesto de que las preferencias de cada hospital sobre grupos de estudiantes "responden" a una preferencia estricta sobre estudiantes individuales. En este contexto, se estudia el juego de revelación de preferencias inducido por la asignación de estudiantes a hospitales mediante la versión del mecanismo de aceptación diferida en el que los estudiantes hacen propuestas a los hospitales. El resultado principal es que la aciclicidad del perfil de preferencias de los hospitales (Romero-Medina y Triossi, 2013a) es una condición necesaria y suficiente para garantizar la estabilidad de cualquier asignación que resulte de un equilibrio de Nash en el que la estrategia de cada hospital consista solamente en eliminar estudiantes de su verdadera lista de preferencias.

**Palabras Clave:** Asignaciones, estabilidad, aciclicidad, estrategias de eliminación, equilibrio de Nash

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\*This article was originally published by The B.E. Journal of Theoretical Economics. The final publication is available at [www.degruyter.com](http://www.degruyter.com), DOI: <https://doi.org/10.1515/bejte-2016-0131>. Financial support from the Consejo Nacional de Ciencia y Tecnología (CONACyT), AGAUR-Generalitat de Catalunya, Universitat Autònoma de Barcelona through PIF grant 41201-9/2010 and the Spanish Ministry of Economy and Competitiveness through FPI grant BES-2012-055341 is gratefully acknowledged.

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# 1 Introduction

We study two-sided many-to-one matching markets that employ the student-proposing deferred acceptance (DA) mechanism. The DA mechanism is based on Gale and Shapley’s (1962) deferred acceptance algorithm and is used in a number of real-life matching markets. Two prominent examples are the New York City high-school assignment system and the National Resident Matching Program (NRMP). The latter is a clearinghouse that matches medical students to hospitals in the U.S.<sup>1</sup> In view of this application, we refer to agents on the “many” side of the market as students and to agents on the “one” side of the market as hospitals.

An important property of matchings is stability. Stability requires that there is no individual agent who prefers to become unmatched nor a pair of agents who prefer to be assigned to each other (possibly in replacement of their current partners). Theoretically, stable matchings are robust to rematching. Empirically, there is evidence that in centralized labor markets, clearinghouses that employ stable mechanisms, i.e., mechanisms that select stable matchings with respect to reported preferences, are more often successful than clearinghouses that employ unstable mechanisms.<sup>2</sup>

In practice, the DA mechanism asks for preferences over individual partners. So, it induces a preference revelation game for agents. Dubins and Freedman (1981) show that in the game induced by DA, it is a weakly dominant strategy for each student to report his true preferences over (individual) hospitals. In light of this result, we assume throughout that students are truthful. It is worth noting that the assumption that students play their weakly dominant strategy of truth-telling is made in most of the literature, for example, Roth (1984b), Kojima and Pathak (2009), Ma (2010), and Jaramillo, Kayi and Klijn (2013). Hospitals, on the other hand, can sometimes benefit from misrepresenting their preferences (see Roth and Sotomayor; 1990, Example 4.1). A concern in the operation of real-life matching markets that employ the DA mechanism is that preference misrepresentation by hospitals can lead to outcomes that are not stable with respect to the agents’ true preferences.

When each hospital has a single position, the set of Nash equilibria outcomes of the game induced by DA coincides with the set of stable matchings (Gale and Sotomayor, 1985 and Roth, 1984b). Therefore, although misrepresentations by hospitals are possible, unstable equilibrium outcomes are not a concern in one-to-one matching markets. Unfortunately, when hospitals have multiple positions, equilibrium outcomes can be unstable (Roth and Sotomayor; 1990, Corollary 5.17).

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<sup>1</sup>See Roth and Peranson (1999) for details on the design of the NRMP.

<sup>2</sup>See Roth (2002) for a comparison between stable and unstable mechanisms used in practice.

We consider the case where each hospital has multiple positions and responsive preferences over sets of students.<sup>3</sup> Even under responsiveness, equilibrium outcomes can be unstable (Roth and Sotomayor; 1990, Corollary 5.17). In this paper, we show that the “acyclicity” of the hospitals’ preference profile, a condition introduced by Romero-Medina and Triossi (2013a), is necessary and sufficient to ensure that the outcome of any “dropping equilibrium” is stable with respect to the true preferences. In what follows, we explain the concepts of acyclicity and dropping equilibrium.

To describe acyclicity consider an analog clock and let each number on the clock represent a different student. The hospitals’ preference profile has a cycle (of length twelve) if hospital one prefers student one to student two, hospital two prefers student two to student three and so on. Lastly, hospital twelve prefers student twelve to student one. The hospitals’ preference profile is acyclic if it has no cycles of any length.

Acyclicity ensures that the set of stable matchings is a singleton in one-to-one matching markets (Romero-Medina and Triossi, 2013a) and in many-to-one matching markets (Akahoshi, 2014). In practice, acyclic preference can arise if, for example, all students are ranked according to some criteria such as test scores.<sup>4</sup> If hospitals rely on such criterion, then individual students are ranked in the same way by each hospital (and hence acyclicity is satisfied), although the rankings of sets of students are not necessarily the same for every hospital.

A dropping strategy is obtained from a hospital’s true preference list by making some acceptable students unacceptable, i.e., the order of any pair of acceptable students in the hospital’s submitted list is not reversed with respect to its true preferences. The class of dropping strategies is strategically exhaustive: fixing the other hospitals’ strategies, the match obtained from any strategy can be replicated or improved upon by a dropping strategy (Kojima and Pathak; 2009, Lemma 1). In other words, any hospital can obtain the same or a better group of students by playing a dropping strategy. The exhaustiveness property makes very plausible the use of dropping strategies. Therefore, focusing on equilibria where each hospital plays a dropping strategy is reasonable. We call this type of equilibrium a dropping equilibrium.

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<sup>3</sup>A hospital’s preferences are responsive if (i) faced with two sets of students that differ only in one student, the hospital prefers the set of students containing the more preferred student and (ii) as long as the hospital has unfilled positions, it prefers to fill a position with an “acceptable” student rather than leaving it unfilled. We give a formal definition in the next section.

<sup>4</sup>While this is not the case of the NRMP, test scores are used as the single criterion to rank students in other applications. One example is the high school admissions system in Mexico City (see Chen and Pereyra, 2017).

Dropping equilibria always exist. In particular, any stable matching can be obtained as the outcome of a dropping equilibrium (Jaramillo, Kayı and Klijn; 2013, Proposition 1). As stated before, a reason to restrict our analysis to dropping equilibria is that the use of dropping strategies is appealing to hospitals given their simplicity and their exhaustiveness property. However, there are Nash equilibrium outcomes that cannot be obtained as the outcome of a dropping equilibrium (Jaramillo, Kayı and Klijn; 2013, Example 1). Therefore, while focusing on dropping equilibria is reasonable, it is not an exhaustive analysis of equilibrium.

Our sufficiency result is as follows. Suppose that the hospitals' preference profile is acyclic. Then, the outcome of any dropping equilibrium is stable with respect to the true preferences. Intuitively, this means that when the hospitals' preference profile is acyclic and each hospital plays a dropping strategy, we can expect the DA mechanism to deliver a stable outcome even if some hospitals are dishonest about their true preferences.

The restriction to dropping equilibria is crucial for the result. In fact, in Example 1, we present a market where (i) the hospitals' preference profile is acyclic, (ii) there is an equilibrium in which one hospital does not play a dropping strategy, and (iii) the equilibrium outcome is unstable.

We finish with the following necessity result: if the hospitals' preference profile has a cycle, then it is impossible to ensure that the outcome of every dropping equilibrium is stable for each profile of students' preferences and each vector of hospitals' capacities.

Ma (2010) studies the same preference revelation game. He focuses on a subclass of dropping strategies, the so-called truncation strategies. He shows that the outcome of any equilibrium in which each hospital plays a truncation strategy is either unstable or coincides with the hospital optimal-stable matching. Truncation strategies are not strategically exhaustive (Kojima and Pathak, 2009). Moreover, "truncation equilibria" do not always exist (Ma, 2010).

Romero-Medina and Triossi (2013b) show that acyclicity is a necessary and sufficient condition to ensure the stability of Nash equilibria outcomes in capacity reporting games. They also consider generalized games of manipulation in which hospitals move first and state their capacities, and students are then assigned to hospitals using a stable mechanism. In the latter setting, they show several results about the implementation of stable matchings.

## **2 Model**

We borrow notation from Jaramillo, Kayı and Klijn (2013).

There are two finite sets of **students**  $S$  and of **hospitals**  $H$ . Let  $I = S \cup H$  be the set of agents. We denote a generic student, hospital, and agent by  $s$ ,  $h$  and  $i$  respectively. For each hospital  $h$ , there is an integer **capacity**  $q_h \geq 1$  that represents the number of positions it offers. Student  $s$  can work for at most one hospital and hospital  $h$  can hire at most  $q_h$  students. Let  $q = (q_h)_{h \in H}$ .

Each students  $s$  has a complete, transitive, and strict **preference relation**  $P_s$  over the hospitals and the prospect of “being unmatched” (or some outside option), which is denoted by  $\emptyset$ . For  $h, h' \in H \cup \{\emptyset\}$ , we write  $h P_s h'$  if student  $s$  prefers  $h$  to  $h'$  ( $h \neq h'$ ), and  $h R_s h'$  if  $s$  finds  $h$  at least as good as  $h'$ , i.e.,  $h P_s h'$  or  $h = h'$ . If  $h \in H$  is such that  $h P_s \emptyset$ , then we call  $h$  an **acceptable hospital** for student  $s$ . We represent the preferences of students over hospitals by a list of hospitals ordered from the most preferred to the worst preferred in line from left to right. For example,  $P_s : h, h', \emptyset$ , means that  $h$  is  $s$ 's most preferred hospital,  $h'$  is  $s$ 's second most preferred hospital and any other hospital is unacceptable to  $s$ . Let  $P_S = (P_s)_{s \in S}$ .

Let  $h \in H$ . A subset of students  $S' \subseteq S$  is **feasible** for hospital  $h$  if  $|S'| \leq q_h$ . Let  $\mathcal{F}_h = \{S' \subseteq S : |S'| \leq q_h\}$  denote the collection of feasible subsets of students for hospital  $h$ . The element  $\emptyset \in \mathcal{F}_h$  denotes “being unmatched” (or some outside option). Hospital  $h$  has a complete, transitive, and strict **preference relation**  $\succ_h$  over  $\mathcal{F}_h$ . For  $S', S'' \in \mathcal{F}_h$  we write  $S' \succ_h S''$  if hospital  $h$  prefers  $S'$  to  $S''$  ( $S' \neq S''$ ) and  $S' \succeq_h S''$  if hospital  $h$  finds  $S'$  at least as good as  $S''$ , i.e.,  $S' \succ_h S''$  or  $S' = S''$ . Let  $\succ = (\succ_h)_{h \in H}$ .

Let  $P_h$  be the restriction of  $\succ_h$  to  $\{\{s\} : s \in S\} \cup \{\emptyset\}$ , i.e., individual students in  $S$  and being unmatched. For  $s, s' \in S \cup \{\emptyset\}$ , we write  $s P_h s'$  if  $s \succ_h s'$ , and  $s R_h s'$  if  $s \succeq_h s'$ .<sup>5</sup> Let  $\mathcal{P}_h$  be the set of all such restrictions for hospital  $h$ . Agent  $s \in S$  is an **acceptable student** for a hospital  $h$  with preferences relation  $P_h$  if  $s P_h \emptyset$ . We represent the preferences of a hospital over individual students by a list of students ordered from the most preferred to the worst preferred in line from left to right. For example,  $P_h : s, s', \emptyset$ , means that  $s$  is  $h$ 's most preferred student,  $s'$  is  $h$ 's second most preferred student and any other student is unacceptable to  $h$ . Let  $P_H = (P_h)_{h \in H}$  and  $P = (P_S, P_H)$ . Finally, for  $H' \subseteq H$ , let  $P_{H'} = (P_{h'})_{h' \in H'}$  and  $P_{-H'} = (P_h)_{h \in H \setminus H'}$ .

We assume that for each hospital  $h$ ,  $\succ_h$  is **responsive**, or more precisely, a responsive extension of  $P_h$ , i.e., for each  $S' \in \mathcal{F}_h$ , (i) if  $s \in S \setminus S'$  and  $|S'| < q_h$ , then  $(S' \cup s) \succ_h S'$  if and only if  $s P_h \emptyset$  and (ii) if  $s \in S \setminus S'$  and  $s' \in S'$ , then  $((S' \setminus s') \cup s) \succ_h S'$  if and only if  $s P_h s'$ .

A **(many-to-one) market** is given by  $(S, H, P_S, \succ_H, q)$  or, when no confusion is possible,  $(P_S, \succ_H$

<sup>5</sup>With some abuse of notation we often write  $x$  for a singleton  $\{x\}$ .

) for short. Let  $(P_S, \succ_H)$  be a market. A **matching** is a function  $\mu$  on the set  $S \cup H$  such that (1) each student is either matched to exactly one hospital or unmatched, i.e., for all  $s \in S$  either  $\mu(s) \in H$  or  $\mu(s) = s$ ; (2) each hospital is matched to a feasible set of students, i.e., for all  $h \in H$ ,  $\mu(h) \in \mathcal{F}_h$  and (3) a student is matched to a hospital if and only if the hospital is matched to the student, i.e., for all  $s \in S$  and  $h \in H$ ,  $\mu(s) = h$  if and only if  $s \in \mu(h)$ .

A matching  $\mu$  is individually rational if no agent would be better off by breaking a current match. Formally, a matching  $\mu$  is **individually rational** if for each  $s \in S$  and each  $h \in H$ , if  $\mu(s) = h$ , then  $h P_s s$  and  $s P_h \emptyset$ .

A student-hospital pair  $(s, h)$  is a **blocking pair** for  $\mu$  if (1)  $h P_s \mu(s)$ , and (2)  $[|\mu(h)| < q_h$  and  $s P_h \emptyset]$  or there is  $s' \in \mu(h)$  such that  $s P_h s'$ . A matching is **stable** if it is individually rational and there are no blocking pairs. It is well-known that in many-to-one matching with responsive preferences, stability coincides with group stability, which excludes the existence of any larger blocking coalition as well (see Roth and Sotomayor, 1990, Lemma 5.5). Since stability does not depend on the particular responsive extensions of the agent's preferences over individual acceptable partners, we denote a market only by a preference profile  $P$ .

A mechanism assigns a matching to each market. We assume that capacities are commonly known by the agents (because, for instance, capacities are determined by law). Therefore, the only information that the mechanism asks from the agents are their preferences over the other side of the market. Many real-life centralized matching markets only ask for the preferences  $P = (P_i)_{i \in I}$  over individual partners, i.e., they do not depend on the particular responsive extensions. In this paper we focus on the **student-proposing deferred acceptance mechanism**,  $DA$ , which is based on the Gale and Shapley's (1962) deferred acceptance algorithm. Let  $P = (P_S, P_H)$  be a profile (of preferences over individual agents). Then, the outcome of  $DA$  is denoted by  $DA(P)$  and it is computed as follows.

Step 1: Each student  $s$  proposes to the hospital that is ranked first in  $P_s$  (if there is such hospital then  $s$  remains unmatched). Each hospital  $h$  considers its proposers and tentatively assigns its  $q_h$  positions to these students one at a time following the preferences  $P_h$ . All other proposers are rejected.

Step  $k$ ,  $k \geq 2$ : Each student  $s$  that is rejected in Step  $k - 1$  proposes to the next hospital in his list  $P_s$  (if there is no such hospital then  $s$  remains single). Each hospital  $h$  considers the students that were tentatively assigned a position at  $h$  in Step  $k - 1$  together with its new proposers. Hospital  $h$  tentatively assigns its  $q_h$  positions to these students one at a time following the preferences  $P_h$ . All other proposers are rejected.

The algorithm terminates when no student is rejected. Then, all tentative matches become final and  $DA(P)$  is the resulting matching. Gale and Shapley (1962, p.14) proved that  $DA(P)$  is stable with respect to  $P$ . Moreover,  $DA(P)$  is called the student-optimal stable matching since it is the best (worst) stable matching for the students (hospitals) with respect to  $P$  (Gale and Shapley, 1962, Theorem 2 and Roth and Sotomayor, 1990, Corollary 5.32).

Under  $DA$ , it is a weakly dominant strategy for the students to reveal their true preferences (Roth, 1985, Theorem 5\*). Since we focus on  $DA$ , we assume that students are truthful and that hospitals are the only strategic agents. Henceforth we fix and suppress  $P_S$ .

A **strategy** is an (ordered) preferences list of a subset of students. More precisely, for each hospital  $h$ ,  $\mathcal{P}_h$  is the set of strategies and  $\mathcal{P} \equiv \times_{h \in H} \mathcal{P}_h$  is the set of strategy profiles,  $DA$  is the outcome function, and the outcome is evaluated through the (true) preference relations  $\succ_H$ . A profile of strategies  $Q$  is a **Nash equilibrium** of the **game**  $(H, \mathcal{P}, DA, \succ_H)$  if for each hospital  $h$  and each strategy  $Q'_h$ ,  $DA(Q)(h) \succeq_h DA(Q'_h, Q_{-h})(h)$ . When no confusion is possible, a game  $(H, \mathcal{P}, DA, \succ_H)$  is denoted by  $\succ_H$ . A result due to Roth (1982, Theorem 3) implies that submitting its true preferences is in general not a weakly dominant strategy for a hospital.

A dropping strategy of a hospital is an ordered list obtained from its true ordered list of acceptable students by removing some acceptable students, i.e., the order of any pair of students in the hospital's submitted list is not reversed with respect to its true preferences (Kojima and Pathak, 2009). Formally, for a hospital  $h$  with preferences  $P_h$  over individual students,  $P'_h$  is a **dropping strategy** if for any students  $s, s' \in S$ , [if  $s R'_h s' R'_h \emptyset$ , then  $s R_h s' R_h \emptyset$ ]. Finally, a **dropping equilibrium** is a Nash equilibrium in which each hospital plays a dropping strategy.

We define an acyclicity condition on profiles of hospitals' preferences introduced by Romero-Medina and Triossi (2013a).

A profile of hospitals' preference relations,  $P_H$ , has a **cycle** of length  $l \geq 2$  if there are  $l$  distinct hospitals  $h_1, h_2, \dots, h_l \in H$  and  $l$  distinct students  $s_1, s_2, \dots, s_l \in S$  such that for each  $i \in \{1, 2, \dots, l\}$ ,  $s_{i+1} P_{h_i} s_i P_{h_i} \emptyset$ , where  $s_{l+1} \equiv s_1$ . If  $P_H$  has no cycle, it is **acyclic**.

If there is a cycle of length  $l$ , it is denoted by  $\{P_{h_1}, P_{h_2}, \dots, P_{h_l}; s_1, s_2, \dots, s_l\}$ .

When each hospital's preferences are responsive, the acyclicity of the hospitals' preference profile ensures that the set of stable matchings is a singleton (Romero-Medina and Triossi, 2013a and Akahoshi, 2014).

### 3 Results

In this section, we present two main results. Our first (main) result is that if the hospitals' preference profile is acyclic, then any dropping equilibrium outcome is stable. When students and hospitals are truthful, we can be confident that the outcome of the DA mechanism is going to be stable. However, hospitals can sometimes benefit by misrepresenting their preferences and by doing so, they undermine the stability of the DA outcome. Intuitively, our first result establishes that when the profile of hospitals preferences is acyclic and each hospital plays a dropping strategy, we can expect the outcome of DA to be stable even if hospitals are dishonest about their true preferences. In this sense, we provide theoretical support for the well functioning of certain matching markets in practice.

We start by stating the following useful lemma.

**Lemma 1.** Let  $\tilde{Q}$  be a dropping equilibrium of the game  $\succ_H$ . Suppose  $\mu = DA(\tilde{Q})$  is blocked by a pair  $(s', h')$  at  $P$ . Then, there is  $\mu'$  such that

- (A)  $\mu'(h') \neq \mu(h')$
  - (B)  $|\mu(s)| = |\mu'(s)|$  for each  $s \in S$  and  $|\mu(h)| = |\mu'(h)|$  for each  $h \in H$ .
- Moreover, for each  $h \in H$  with  $\mu(h) \neq \mu'(h)$ ,
- (C) if  $h \neq h'$ , then  $|\mu(h)| = |\mu'(h)| = q_h$ .
  - (D) if  $h \neq h'$ , then  $\mu(h) \setminus \mu'(h) \neq \emptyset$  and  $\mu'(h) \setminus \mu(h) \neq \emptyset$ .
  - (E) for each  $r \in \mu(h)$  and  $s \in \mu'(h) \setminus \mu(h)$ ,  $r \tilde{Q}_h s$ .

The proof of Lemma 1 is in the Appendix.

Now we are ready to state and prove our main result.

**Theorem 1.** Assume  $P_H$  is acyclic. Let  $Q$  be a dropping equilibrium of the game  $\succ_H$ . Then,  $DA(Q)$  is stable at  $P$ .

**Proof:** Let  $\mu = DA(Q)$ . Since  $Q$  is an equilibrium,  $\mu$  is individually rational at  $P$ . Assume by contradiction that  $\mu$  is blocked by some pair  $(s', h')$  at  $P$ . By Lemma 1 there is  $\mu'$  that satisfies (A), (B), (C), (D) and (E).

Apply the following algorithm based on an algorithm in Akahoshi (2014, page 11) to  $\mu$  and  $\mu'$  and obtain sequences of hospitals and students.

**Algorithm 1.**<sup>6</sup>

**Step 1.** Let  $h_1 \equiv h'$  and  $s_2 \in \mu(h_1) \setminus \mu'(h_1)$ . Define  $h_2 \in H \setminus \{h_1\}$  by  $h_2 \equiv \mu'(s_2)$ .

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<sup>6</sup>Algorithm 1 is based on the algorithm described by Akahoshi (2014) in page 11, paragraph 4.

**Step** ( $k \geq 2$ ): Choose  $s_{k+1} \in S$  such that  $s_{k+1} \in \mu(h_k) \setminus \mu'(h_k)$  and  $h_{k+1} \in H \setminus \{h_k\}$  such that  $h_{k+1} \equiv \mu'(s_{k+1})$ . If  $h_{k+1} \in \{h_1, h_2, \dots, h_{k-1}\}$ , then the algorithm terminates. If not, go to the next step.

**Output:** If the algorithm terminates at Step  $l \geq 2$  with  $h_{l+1} = h_j$  ( $j \geq 1$ ), let the output be given by the hospitals  $\{h_j, h_{j+1}, \dots, h_l\}$  and the students  $\{s_{j+1}, s_{j+2}, \dots, s_{l+1}\}$ .  $\diamond$

We first show that the algorithm is well-defined. The existence of  $s_2$  and  $h_2$  follows from Lemma 1 (A) and (B). At step  $k$ , since  $s_k \in \mu'(h_k) \setminus \mu(h_k)$ , we can take  $h_{k+1} \in H$  such that  $h_{k+1} = \mu'(s_{k+1})$  and  $h_{k+1} \neq h_k$  by Lemma 1 (B). Since  $H$  is finite, the algorithm terminates in finitely many steps.

By construction, when the algorithm terminates at step  $l$ , the output satisfies the following:

$$\begin{array}{ll} s_{j+1} \in \mu(h_j) \setminus \mu'(h_j) & s_{l+1} \in \mu'(h_j) \setminus \mu(h_j) \\ s_{j+2} \in \mu(h_{j+1}) \setminus \mu'(h_{j+1}) & s_{j+1} \in \mu'(h_{j+1}) \setminus \mu(h_{j+1}) \\ \vdots & \vdots \\ s_{l+1} \in \mu(h_l) \setminus \mu'(h_l) & s_l \in \mu'(h_l) \setminus \mu(h_l) \end{array}$$

We show that  $Q_H$  has a cycle  $\{Q_{h_j}, Q_{h_{j+1}}, \dots, Q_{h_l}; s_{l+1}, s_{j+1}, s_{j+2}, \dots, s_l\}$ . Since  $h_j, h_{j+1}, \dots, h_l$  are all distinct, so are  $s_{j+1}, s_{j+2}, \dots, s_{l+1}$ . For each  $k \in \{j+1, j+2, \dots, l\}$ ,  $s_k \in \mu'(h_k)$  and  $s_{k+1} \in \mu(h_k)$ . Furthermore,  $s_{l+1} \in \mu'(h_j)$  and  $s_{j+1} \in \mu(h_j)$ . Therefore, by Lemma 1 (E),  $s_{k+1} Q_{h_k} s_k Q_{h_k} \emptyset$  for each  $k \in \{j+1, j+2, \dots, l\}$ , and  $s_{j+1} Q_{h_j} s_{l+1} \tilde{Q}_{h_j} \emptyset$ . Hence,  $\{Q_{h_j}, Q_{h_{j+1}}, \dots, Q_{h_l}; s_{l+1}, s_{j+1}, s_{j+2}, \dots, s_l\}$  is a cycle for  $Q_H$ .<sup>7</sup>  $\blacktriangle$

Since  $Q_H$  consists of dropping strategies, the cycle in  $Q_H$  is also a cycle in  $P_H$ . This is a contradiction to the acyclicity of  $P_H$ .  $\blacksquare$

When hospitals have responsive preferences, acyclicity is a sufficient condition for the set of stable matchings to be a singleton (see Romero-Medina and Triossi, 2013a and Akahoshi, 2014). Thus, the next result is obtained as an immediate corollary to Theorem 1 in this paper.

**Corollary 1.** Assume  $P_H$  is acyclic. Let  $Q$  be a dropping equilibrium of the game  $\succ_H$ . Then,  $DA(Q)$  is the unique stable matching at  $P$ . In particular,  $DA(Q) = DA(P)$ .

**Example 1. (Unstability of non dropping equilibrium.)**

Jaramillo, Kayı and Klijn (2013) present the following many-to-one market with 4 students, 2 hos-

<sup>7</sup>The construction process of cycles in this proof is very similar to that in Akahoshi (2014, page 11).

pitals, and preferences over individual partners  $P$  given by the columns in Table 1.<sup>8</sup> Both hospitals have capacity 2.

Table 1: Preferences  $P$  in Example 1

Hospitals		Students			
$P_{h_1}$	$P_{h_2}$	$P_{s_1}$	$P_{s_2}$	$P_{s_3}$	$P_{s_4}$
$s_1$	$s_4$	$h_2$	$h_1$	$h_1$	$h_1$
$s_2$	$s_1$	$h_1$	$h_2$	$h_2$	$h_2$
$s_3$	$s_2$				
	$s_3$				

Jaramillo, Kayı and Klijn (2013) use this market to show that there are equilibrium outcomes that cannot be obtained as the outcome of a dropping equilibrium. We use this same market to show that even if  $P_H$  is acyclic, there can be equilibria in which at least one hospital does not play a dropping strategy (non dropping equilibrium), and the outcome it induces is unstable.

To check for the acyclicity of  $P_H$ , it is enough to note that (i) the set of students that are acceptable to both hospitals  $h_1$  and  $h_2$  is  $\{s_1, s_2, s_3\}$ , and that (ii) both hospitals rank  $s_1, s_2, s_3$  in the same order.

Furthermore, from Jaramillo, Kayı and Klijn (2013), we know the following about market  $P$ :

(1) There is a unique stable matching for  $P$ , and it is given by

$$DA(P) : \begin{array}{cc} h_1 & h_2 \\ | & | \\ \{s_2, s_3\} & \{s_1, s_4\} \end{array}$$

which is the boxed matching in Table 1.

(2) The strategy profile  $Q = (Q_{h_1}, Q_{h_2})$ , where  $Q_{h_1} : s_1, s_2, s_4, s_3$  and  $Q_{h_2} : s_4, s_2, s_3$  for hospitals  $h_1$  and  $h_2$  is a Nash equilibrium, and it induces the matching

<sup>8</sup>For example, the first column of Table 1 depicts the preference  $P_{h_1} : s_1, s_2, s_3$ . Note that student  $s_4$  is not acceptable for hospital  $h_1$ .

$$\begin{array}{cc}
& h_1 & h_2 \\
DA(Q) : & | & | \\
& \{s_1, s_2\} & \{s_3, s_4\}
\end{array}$$

which is the boldface matching in Table 1.

(3)  $DA(Q)$  cannot be obtained in any equilibrium that consist of dropping strategies.

Clearly,  $Q_{h_1}$  is not a dropping strategy. Moreover, since  $DA(Q) \neq DA(P)$ ,  $DA(Q)$  is a unstable matching.

This example shows that acyclicity cannot ensure the stability of non dropping equilibrium outcomes.

The existence of a cycle in the hospitals' preference profile makes it impossible to ensure the stability of dropping equilibrium outcomes for every profile of students' preferences and every vector of capacities. That is, acyclicity of hospitals' preference profile is necessary to ensure the stability of dropping equilibrium outcomes. This result implies that if a single cycle is identified in the hospitals' preference profile, then there is no guarantee that the outcome of DA will be stable.

**Proposition 1.** Assume that  $P_H$  has a cycle. Then, there are preferences  $P_s$  for each  $s \in S$ , a capacity  $q_h$  and a dropping strategy  $Q_h$  for each  $h \in H$  such that the profile  $Q_H$  is a Nash equilibrium of the game  $\succ_H$  and  $DA(P_S, Q_H)$  is not stable.

The proof to Proposition 1 is relegated to the Appendix.

**Example 2. (A market with an unstable dropping equilibrium outcome.)**

Consider a market with 2 hospitals and 3 students. Let the hospitals' preference profile  $P_H$  be given by the columns in Table 2.

Table 2: A hospitals' preference profile with a cycle

Hospitals	
$P_{h_1}$	$P_{h_2}$
$s_1$	$s_2$
$s_2$	$s_1$
$s_3$	$s_3$

Since  $s_1 P_{h_1} s_2 P_{h_1} \emptyset$  and  $s_2 P_{h_2} s_1 P_{h_2} \emptyset$ , the hospitals' preference profile  $P_H$  has a cycle  $\{P_{h_1}, P_{h_2}; s_1, s_2\}$ . By Proposition 1, it is possible to find a preference  $P_{s_i}$  for each  $i = 1, 2, 3$ ; a capacity  $q_{h_i}$  and a dropping strategy  $Q_{h_i}$  for each  $i = 1, 2$ , such that  $Q_H = (Q_{h_1}, Q_{h_2})$  is an equilibrium and  $DA(P_S, Q_H)$  is unstable.

We let hospitals' capacities be  $q_{h_1} = 2$  and  $q_{h_2} = 1$  and student's preferences be as in Table 3.

Table 3: Preferences  $P$  in Example 2

Hospitals		Students		
$P_{h_1}$	$P_{h_2}$	$P_{s_1}$	$P_{s_2}$	$P_{s_3}$
$s_1$	$s_2$	$h_2$	$h_1$	$h_1$
$s_2$	$s_1$	$h_1$	$h_2$	$h_2$
$s_3$	$s_3$			

Consider the following dropping strategies  $Q_{h_1} : s_1, s_3$  for hospital  $h_1$ , and  $Q_{h_2} = P_{h_2} : s_2, s_1, s_3$  for hospital  $h_2$ . Routine computations<sup>9</sup> show that  $Q_H = (Q_{h_1}, Q_{h_2})$  is a Nash equilibrium and it induces the matching

$$DA(P_S, Q_H) : \begin{array}{cc} & h_1 & h_2 \\ & | & | \\ & \{s_1, s_3\} & \{s_2\} \end{array}$$

which is the boxed matching in Table 3.

Finally, note that the matching  $DA(P_S, Q_H)$  is unstable as it is blocked by  $s_2$  and  $h_1$ .

## 4 Final Remarks

Roth (2002) documents that in matching markets in practice, stable mechanisms are more often successful than unstable ones. For example, unstable mechanisms for regional medical markets in Birmingham, Newcastle or Sheffield were replaced while several other stable mechanisms like the medical

<sup>9</sup>From Kojima and Pathak (2009, Lemma 1) it follows that it is enough to only consider dropping strategies for possible profitable deviations.

regional market in Cardiff, the NRMP or dental residencies in the U.S. are still in use (See Roth, 2002, for more details). This is so despite the fact that stable mechanism like the the DA mechanism are manipulable or their equilibrium outcomes can be unstable. In this paper, we provide further theoretical evidence as to why markets that employ DA may perform well in real-life applications. We show that when each hospital plays a dropping strategy, acyclicity is a necessary and sufficient condition to ensure that equilibrium outcomes are stable. A question that remains open is to provide a condition that ensures the stability of equilibrium outcomes when no restriction on the type of strategies that hospitals can play is imposed.

## References

- Akahoshi, T. (2014). Singleton core in many-to-one matching problems. *Mathematical Social Sciences* 72, 7–13.
- Chen, L. and J. S. Pereyra (2017). Self-selection in school choice. *Working paper*.
- Dubins, L. E. and D. A. Freedman (1981). Machiavelli and the Gale-Shapley algorithm. *American Mathematical Monthly* 88(7), 485–494.
- Erdil, A. and H. Ergin (2008). What’s the matter with tie-breaking? Improving efficiency in school choice. *American Economic Review* 98(3), 669–689.
- Gale, D. and L. S. Shapley (1962). College admissions and the stability of marriage. *American Mathematical Monthly* 69(1), 9–15.
- Gale, D. and M. A. O. Sotomayor (1985). Ms Machiavelli and the stable matching problem. *American Mathematical Monthly* 92(4), 261–268.
- Jaramillo, P., Ç Kayi, and F. Klijn (2013). Equilibria under deferred acceptance: Dropping strategies, filled positions, and welfare. *Games and Economic Behavior* 82(1), 693–701.
- Kojima, F. and P. A. Pathak (2009). Incentives and stability in large two-sided matching markets. *American Economic Review* 99(3), 608–627.
- Ma, J. (2010). The singleton core in the hospital-admissions problem and its application to the national resident matching program (NRMP). *Games and Economic Behavior* 60(01), 150–164.
- Romero-Medina, A. and M. Triossi (2013a). Acyclicity and singleton cores in matching markets. *Economics Letters* 118(1), 237–239.
- Romero-Medina, A. and M. Triossi (2013b). Games with capacity manipulation: Incentives and nash equilibria. *Social Choice and Welfare* 41(3), 701–720.
- Roth, A. E. (1982). The economics of matching: Stability and incentives. *Mathematics of Operations Research* 7(4), 617–628.
- Roth, A. E. (1984a). The evolution of the labor market for medical interns and residents: A case study in game theory. *Journal of Political Economy* 92(6), 991–1016.

- Roth, A. E. (1984b). Misrepresentation and stability in the marriage problem. *Journal of Economic Theory* 34(2), 383–387.
- Roth, A. E. (1985). The hospital admission problem is not equivalent to the marriage problem. *Journal of Economic Theory* 36(2), 277–288.
- Roth, A. E. (2002). The economist as engineer: game theory, experimentation as tools for design economics. *Econometrica* 70, 1341–1378.
- Roth, A. E. and E. Peranson (1999). The redesign of the matching market for american physicians: Some engineering aspects of economic design. *American Economic Review* 89(4), 748–780.
- Roth, A. E. and M. A. O. Sotomayor (1990). *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Econometric Society Monograph Series. New York: Cambridge University Press.

# Appendix

## Proof of Lemma 1

In order to prove Lemma 1, we introduce the following notation. For every integer  $k \geq 1$ , let  $X(Q, h, k)$  be the set of students that will have proposed to hospital  $h$  by step  $k$  under  $DA(Q)$ , i.e., in some step  $l \in \{1, \dots, k\}$  of  $DA(Q)$ , and let  $m(Q, h, k)$  be the set of students tentatively matched to  $h$  in some step  $l \in \{1, \dots, k\}$  of  $DA(Q)$ . Let  $X(Q, h)$  be the set of students that will have proposed to  $h$  by the last step of  $DA(Q)$ , i.e.,  $X(Q, h) = \cup_k X(Q, h, k)$ .

We prove Lemma 1 by proving a series of claims. Let  $Q_{h'}$  list all students in  $\mu(h')$  in the same relative order as in  $P_{h'}$  and report every other student as unacceptable. Let  $Q = (Q_{h'}, \tilde{Q}_{-h'})$ . Let  $Q'_{h'}$  lists all students in  $\mu(h') \cup \{s'\}$  in the same relative order as in  $P_{h'}$  and report every other student as unacceptable. Let  $Q' = (Q'_{h'}, \tilde{Q}_{-h'})$  and let  $\mu' = DA(Q')$ .

**Claim 1.**  $DA(Q) = \mu$ .

**Proof.** Clearly,  $\mu$  is individually rational at  $Q$ . Moreover, if a pair  $(s, h)$  blocks  $\mu$  at  $Q$ , then  $(s, h)$  would also block  $\mu$  at  $P$ . Hence,  $\mu$  is stable at  $Q$ .

Now we show that  $DA(Q)$  is stable at  $\tilde{Q}$ . Clearly,  $DA(Q)$  is individually rational at  $\tilde{Q}$ . Assume that  $s \in S$  and  $h \in H \setminus \{h'\}$  block  $DA(Q)$  at  $\tilde{Q}$ . Since  $Q_s = \tilde{Q}_s$  and  $Q_h = \tilde{Q}_h$ , then  $(s, h)$  would also block  $DA(Q)$  at  $Q$  which is a contradiction to the stability of  $DA$ . So, we conclude that no  $s \in S$  and  $h \in H \setminus \{h'\}$  block  $DA(Q)$  at  $\tilde{Q}$ .

Now we show that no  $s \in S$  and  $h'$  block  $DA(Q)$  at  $\tilde{Q}$ . Since both  $\mu$  and  $DA(Q)$  are stable at  $Q$ , by Roth (1984a),  $|DA(Q)(h')| = |\mu(h')|$ . Moreover, since only the students in  $\mu(h')$  are acceptable to  $h'$  under  $Q_{h'}$ ,  $DA(Q)(h') = \mu(h')$ . Assume by contradiction that  $s \in S$  and  $h'$  block  $DA(Q)$  at  $\tilde{Q}$ . Then,  $[|\mu(h')| < q_{h'} \text{ or } s \tilde{Q}_{h'} s^* \text{ for some } s^* \in \mu(h')] \text{ and } h' P_s DA(Q)(s)$ . Since  $DA(Q)$  and  $\mu$  are stable at  $Q$  and  $DA(Q)$  is the student optimal stable matching at  $Q$ ,  $DA(Q)(s) R_s \mu(s)$ . Hence,  $h' P_s \mu(s)$ . Therefore,  $s$  and  $h'$  also block  $\mu = DA(\tilde{Q})$  at  $\tilde{Q}$ , but this contradicts the stability of  $DA$ . Thus,  $DA(Q)$  is stable at  $\tilde{Q}$ .

Since  $\mu$  is stable at  $Q$ ,  $DA(Q)$  is weakly preferred by all students to  $\mu$ . Moreover, since  $DA(Q)$  is stable at  $\tilde{Q}$ ,  $\mu = DA(\tilde{Q})$  is weakly preferred by all students to  $DA(Q)$ . Therefore,  $\mu = DA(Q)$  as desired. ▲

**Claim 2.** (A) holds, i.e.,  $\mu(h') \neq \mu'(h')$ .

**Proof.** Student  $s'$  proposes to  $h'$  at the same step under  $DA(Q)$  and under  $DA(Q')$ . Furthermore,  $s'$  is tentatively accepted by  $h'$  at  $DA(Q')$ . Since  $s'$  and  $h'$  block  $\mu$  at  $P$ ,  $|\mu(h')| < q_{h'}$  or  $s' P_{h'} s^*$  for some  $s^* \in \mu(h')$ . Since  $Q'_{h'}$  lists students in  $\mu(h') \cup \{s'\}$  according to  $P_{h'}$  and lists any other student as unacceptable,  $s'$  is not rejected in any latter step of  $DA(Q')$ . Thus,  $s' \in \mu'(h')$ , and hence  $\mu'(h') \neq \mu(h')$ .  $\blacktriangle$

**Claim 3.** For each hospital  $h$  and each step  $k$ ,  $X(Q', h, k) \subseteq X(Q, h, k)$ .

**Proof.** For  $k = 1$  the inclusion is in fact an equality since at step 1 of  $DA(Q')$  and  $DA(Q)$  each student proposes to exactly the same hospital.

Assume that the inclusion holds for  $k$ . We will show that the inclusion also holds for  $k + 1$ . Let  $s \in X(Q', h, k + 1)$ . If  $s \in X(Q', h, k)$ , then by induction,  $s \in X(Q, h, k)$  and hence  $s \in X(Q, h, k) \subseteq X(Q, h, k + 1)$ . So, assume  $s \in X(Q', h, k + 1) \setminus X(Q', h, k)$ . Then, in  $DA(Q')$ , student  $s$  proposed to  $h$  at step  $k + 1$  but not at step  $k$ . So,  $s$  was rejected by some hospital  $\bar{h} \neq h$  at step  $k$  of  $DA(Q')$ . By the induction hypothesis,  $s \in X(Q', \bar{h}, k) \subseteq X(Q, \bar{h}, k)$ . If  $\bar{h} \neq h'$ , then  $\bar{h}$  will also have rejected  $s$  by step  $k$  of  $DA(Q)$  since  $Q'_{\bar{h}} = Q_{\bar{h}}$ . Assume  $\bar{h} = h'$ . We consider two cases.

**Case 1.**  $s = s'$  or  $s \notin \mu(h')$ . Then,  $\emptyset Q_{h'} s$ . Thus,  $\bar{h}$  will also have rejected  $s$  by step  $k$  of  $DA(Q)$ .

**Case 2.**  $s \neq s'$  and  $s \in \mu(h')$ . By definition of  $Q'_{h'}$ ,  $s Q'_{h'} \emptyset$ . Since  $s$  is rejected by  $h'$  at step  $k$  of  $DA(Q')$ , (i)  $|m(Q', h', k)| = q_{h'}$  and (ii) for each student  $s^* \in m(Q', h', k)$ ,  $s^* Q'_{h'} s$ . By responsiveness and the fact that  $Q'_{h'}$  is a dropping strategy obtained from  $P_{h'}$ ,  $m(Q', h', k) P_{h'} \mu(h')$ . Moreover, by the definition of  $DA$ ,  $\mu'(h') R_{h'} m(Q', h', k)$ . Hence,  $\mu'(h') P_{h'} \mu(h')$ . This implies that  $Q'_{h'}$  is a profitable deviation for  $h'$  contradicting that  $Q$  is an equilibrium. Therefore,  $s$  is not rejected by  $h'$  at step  $k$  of  $DA(Q')$ . This contradicts the fact that  $s$  is rejected by  $h'$  at step  $k$  of  $DA(Q')$ . Thus, Case 2 is impossible.

Since  $s$  is rejected by  $\bar{h}$  at step  $k$  of  $DA(Q)$  and he makes his proposals in the same order in  $DA(Q)$  and  $DA(Q')$ , he will have proposed to  $h$  by step  $k + 1$  of  $DA(Q)$ . Hence,  $s \in X(Q, h, k + 1)$ .  $\blacktriangle$

We complete the proof. (B) By Claim 3 and the fact that  $\mu' \neq \mu$  (Claim 2),  $\mu'$  is weakly preferred by all students to  $\mu$ . Then, the first part of (B) follows from (i) and (iii) of Lemma 1 in Erdil and Ergin (2008, page 684). The second part of (B) follows from (i) and (ii) of Lemma 1 in Erdil and Ergin (2008, page 684). (C) If  $h \neq h'$  and  $\mu(h) \neq \mu'(h)$ , then by (B) there is  $r \in \mu(h) \setminus \mu'(h)$  and  $s \in \mu'(h) \setminus \mu(h)$  such that  $r, s \in S$ . Moreover, since  $Q_h = Q'_h$ ,  $r$  and  $s$  are acceptable under both  $Q_h$  and  $Q'_h$ . Assume by contradiction that  $|\mu'(h)| < q_h$ . By Claim 3,  $s \in \mu'(h) \subseteq X(Q', h) \subseteq X(Q, h)$ . Therefore,  $s \in \mu(h)$ , but this is a contradiction. Hence,  $|\mu'(h)| = q_h$ . Using (B), we conclude  $|\mu(h)| = |\mu'(h)| = q_h$ . (D) follows from (C) and the fact that  $\mu(h) \neq \mu'(h)$ . (E) Since  $\mu'$  Pareto dominates  $\mu$  for students and  $s \in \mu'(h) \setminus \mu(h)$ ,  $h P_s \mu(s)$ . Assume by contradiction that  $s \tilde{Q}_h r \in \mu(h)$ . Then,  $s$  and  $h$  block  $\mu$  at  $\tilde{Q}$  contradicting the stability of  $DA$ . ■

## Proof of Proposition 1

Assume that  $P_H$  has a cycle of length  $l \geq 2$ , given by

$$\{P_{h_1}, P_{h_2}, \dots, P_{h_l}; s_1, s_2, \dots, s_l\}.$$

We define a preference profile for the students as follows. For each  $i = 1, \dots, l$  let  $P_{s_i} : h_i, h_{i-1}, \emptyset$ , where  $h_0 \equiv h_l$ . For each  $s, s' \in S \setminus \{s_1, \dots, s_l\}$  let  $P_s = P'_s$  and  $\emptyset P_s h$  for each  $h \in \{h_1, \dots, h_l\}$ . That is, all other students have the same preferences and any hospital  $h_1, \dots, h_l$  is unacceptable to them. The preferences of hospitals  $\{h_1, \dots, h_l\}$  and students  $\{s_1, \dots, s_l\}$  are depicted below, vertical dots mean that preferences can be arbitrary, while horizontal dots mean that the pattern of the given sequence continues until it reaches its final element.

$P_{h_1}$	$P_{h_2}$	$\dots$	$P_{h_l}$	$P_{s_1}$	$P_{s_2}$	$\dots$	$P_{s_l}$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$h_1$	$h_2$	$\dots$	$h_l$
$s_2$	$s_3$	$\dots$	$s_1$	$h_l$	$h_1$	$\dots$	$h_{l-1}$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$
$s_1$	$s_2$	$\dots$	$s_l$	$\vdots$	$\vdots$	$\dots$	$\vdots$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$

Let  $q_h = 1$  for all  $h \in H \setminus \{h_1\}$  and  $q_{h_1} = 2$ . Consider the dropping strategy for  $h_1$ ,  $P'_{h_1} : s_2, \emptyset$  and the profile  $P' = (P'_{h_1}, P_{-h_0})$ . Note that at  $P'$  all hospitals play a dropping strategy and all students report their true preferences. Let  $\mu' = DA(P')$ .

We show that for each  $i = 1, \dots, l$ ,  $\mu'(h_i) = \{s_{i+1}\}$ , where  $s_{l+1} \equiv s_1$ . At  $DA(P')$  no student  $s \in S \setminus \{s_1, \dots, s_l\}$  proposes to a hospital in  $\{h_1 \dots h_l\}$ . Furthermore, each student  $s_i$ ,  $i = 1, \dots, l$ , proposes to  $h_i$  at the first step of  $DA(P')$ . Each student  $s_i$ ,  $i = 2, \dots, l$ , is tentatively accepted. However,  $s_1$  is rejected by  $h_1$ . Therefore,  $s_1$  proposes to  $h_l$  at step 2,  $h_l$  tentatively accepts  $s_1$  and rejects  $s_l$ ,  $s_l$  proposes to  $h_{l-1}$  at step 3,  $h_{l-1}$  tentatively accepts  $s_l$  and rejects  $s_{l-1}$ . This process continues until  $s_2$  proposes to  $h_1$ ,  $h_1$  tentatively accepts  $s_2$ . At this point no  $s_i$ ,  $i = 1, \dots, l$ , makes more proposals so the the acceptances become final.

We show that  $P'$  is a Nash equilibrium. Each  $h_i$ ,  $i = 2, \dots, l$  fills its capacity with the best student among the students who find  $h_i$  acceptable. Therefore, no  $h_i$ ,  $i = 2, \dots, l$  has a profitable deviation. To show that  $h_1$  has no profitable deviations either, it is enough to show that  $h_1$  cannot improve by listing  $s_1$  as acceptable. Let  $P''_{h_1}$  be such that  $s_1 P''_{h_1} \emptyset$  and  $P'' = (P''_{h_1}, P_{-h_0})$ . At the first step of  $DA(P'')$  each student  $s_i$ ,  $i = 1, \dots, l$ , proposes to  $h_i$ . Furthermore, each students  $s_i$ ,  $i = 1, \dots, l$ , is tentatively accepted. Therefore, no  $s_i$ ,  $i = 1, \dots, l$ , makes more proposals and the acceptances become final. Since  $DA(P'')(h_1) = \{s_1\}$  and  $s_2 P_{h_1} s_1$ ,  $P''$  is not a profitable deviation. Hence  $P'$  is an equilibrium.

To complete the proof note that since (i)  $h_1 P_{s_1} \mu'(s_1) = h_l$ , (ii)  $|\mu'(h_1)| = 1 < q_{h_1} = 2$  and (iii)  $s_1$  is acceptable to  $h_1$ ,  $h_1$  and  $s_1$  block  $\mu'$  under the true preferences. Thus,  $\mu'$  is not stable with respect to  $P$ . ■